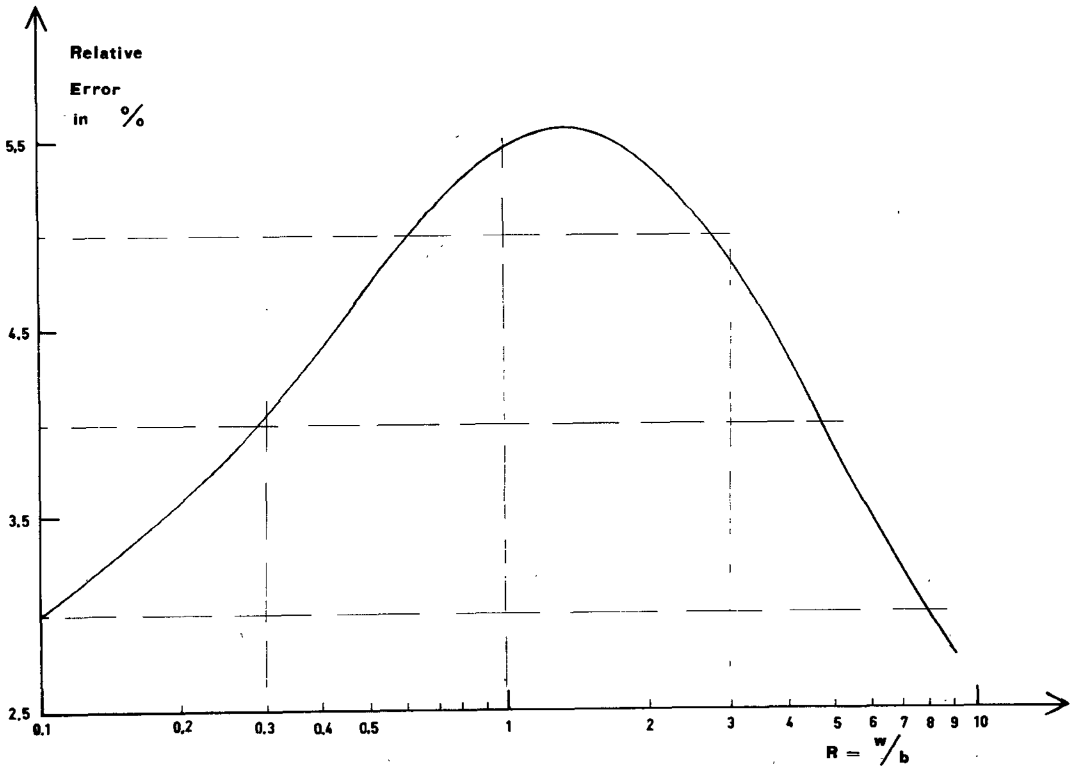

Fig. 2. Error introduced by Rochelle's formula.

important to note that this error is always greater than 3 percent for $0.1 < R < 10$. It is therefore advisable to add a correction coefficient of 1.04 to [footnote 1, eq. (10)] as long as R lies within the interval (0.1,10).

Although, theoretically [footnote 1, eq. (10)] is extremely interesting, the error involved is approximately ten times greater than that of Wheeler. Moreover, it appears that the comparison of [footnote 1, eq. (10)] with Bromwich's formulas (graphs A and B of the article) is not very judicious, since the latter formulas are less accurate than Wheeler's.

The numerical results shown in Table I allow the precision of diverse approximations made on the parallel conductor transmission line to be measured.

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On the Accuracy of the Beam-Wave Theory of the Open Resonator

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In an interesting paper Erickson [1] has demonstrated how perturbation theory can be used to improve the accuracy of the beam-wave theory of the open resonator. Specifically, two defects of beam-wave theory are considered. The first is that the equiphase surfaces of beam-wave theory are not spherical, the second is that the wave function employed is only an approximate solution to the wave equation.

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There is, however, a third defect of an equally fundamental nature, namely, that the boundary condition $u = 0$ over the whole surface of each mirror is not correct for spherical mirrors, if, as is implied, u represents one of the Cartesian components, say E_x, of the transverse electric field. This point has already been considered briefly by Cullen *et al.* [2]. The purpose of the present letter is to demonstrate that this boundary condition error is in fact of comparable importance to the other two defects, at least for the fundamental mode $p = l = 0$.

For this mode, the fractional frequency-shift correction arising from the approximation made in the wave equation is given by Erickson [1, eq. (28)].

$$\frac{\Delta f}{f} = \frac{\lambda}{\pi d} \tan^{-1}\left(\frac{a^2 d}{8k_0}\right). \tag{1}$$

This equation can be written

$$\frac{\Delta f}{f} = \frac{\lambda}{\pi d} \tan^{-1}\left\{\left(\frac{1}{k_0 w_0}\right)^4 \frac{2\pi d}{\lambda}\right\} \simeq 2\left(\frac{1}{k_0 w_0}\right)^4 \tag{2}$$

the approximate form being valid when $(k_0 w_0)^4 \gg 2\pi(d/\lambda)$. Thus the error in the simple beam-wave formula for resonant frequency arising from an approximate wave equation is of the order $(k_0 w_0)^{-4}$. We shall now show that the error due to the use of an incorrect boundary condition is of the same order.

The physical reason why $E_x \neq 0$ on the mirror surface is clear; the electric vector will be normal to the mirror at its surface, and so there will in general be finite components of E_x and E_y on the surface, though these will both vanish on the axis. Suppose u and v represent two different representations of E_x; both satisfy the wave equation, $u = 0$ on S, but $v = v_s$ on S, S being the surface of one of the mirrors. Then the fractional change in frequency arising from the nonvanishing of E_x is

$$\frac{\delta f}{f} = \frac{\int_S (v_s \nabla u) \cdot dS}{k_u^2 \int_V u^2 \, dV} \tag{3}$$

where the surface integral is taken over one of the two mirrors of a symmetrical open resonator, and δf is the frequency that must be added to the frequency of resonance given by simple beam-wave theory.

To calculate v_s approximately, we identify $u = E_x$ and assume $E_y = 0$ for the fundamental TEM_{00q} mode. We then use the divergence equation to estimate E_z, neglecting the variation with z of all but the factor $\exp(-jkz)$ in the expression for E_z. Thus starting with

$$u = E_x = \frac{w_0}{w} \exp\left(-\frac{r^2}{w^2}\right) \cdot \exp - j\left(kz - \tan^{-1} az + \frac{r^2}{w^2} az\right) \tag{4}$$

and using $jkE_z = \partial E_x/\partial x$, we find

$$E_z = j\frac{w_0}{w}\frac{2x}{kw^2}(1 + jaz)\exp\left(-\frac{r^2}{w^2}\right) \cdot \exp - j\left(kz - \tan^{-1} az + \frac{r^2}{w^2} az\right)$$

or

$$E_z = j\frac{2x}{kw^2}\exp\left(-\frac{r^2}{w^2}\right) \cdot \exp - j\left(kz - 2\tan^{-1} az + \frac{r^2}{w^2} az\right). \tag{5}$$

To find standing-wave solutions corresponding to the resonator problem, we can take the sum or difference of E_x and E_x^* and also of E_z and E_z^*. For definiteness, consider the symmetrical solution corresponding to half the sum; the following pair of formulas is found in this way

$$u = E_x = \frac{w_0}{w}\exp\left(-\frac{r^2}{w^2}\right)\cos\left(kz - \tan^{-1} az + \frac{r^2}{w^2} az\right) \tag{4}$$

$$E_z = \frac{2x}{kw^2}\exp\left(-\frac{r^2}{w^2}\right)\sin\left(kz - 2\tan^{-1} az + \frac{r^2}{w^2} az\right). \tag{5}$$

We next represent the spherical mirrors by a parabolic approximation, thus

$$z = z_1 - \frac{r^2}{2R_1} \tag{6}$$

where R_1 is the radius of curvature of the nodal surface of u on the z axis at $z = z_1$. (Note that this deviation from the spherical shape can be allowed for by a separate perturbation calculation in the manner indicated by Erickson.) Using the equation

$$kz_1 - \tan^{-1} az_1 = (q + 1)\frac{\pi}{2} = (2m + 1)\frac{\pi}{2} \tag{7}$$

in which we have used the fact that q must be even, $q = 2m$, say, for the symmetrical modes we are considering, we eventually find for E_2 on the surface of the mirror at $+z_1$ the result

$$E_{z_s} = \frac{2x}{kw_1^2}\exp\left(-\frac{r^2}{w_1^2}\right)(-1)^m\left(\frac{w_0}{w_1}\right). \tag{8}$$

In deriving (8), we have assumed that the variation of w, R, and Φ with z over the mirror surface is negligible; it can be shown that the resulting fractional error in E_{z_s} is of the order $1/k^2w_1^2$ and so can be ignored to first order. To evaluate E_x on the mirror surface, (4) is of course not sufficiently accurate. Instead, we can use the result that $E_{\tan} = 0$ on the mirror surface to express $v_s = E_{x_s}$ in terms of E_{z_s}, thus

$$v_s = E_{z_s}\frac{x}{R_1} = (-1)^m\frac{2x^2}{kw_1^2R_1}\left(\frac{w_0}{w_1}\right)\exp\left(-\frac{r^2}{w_1^2}\right)$$

or

$$v_s = (-1)^m\frac{2r^2\cos^2\phi}{kw_1^2R_1}\left(\frac{w_0}{w_1}\right)\exp\left(-\frac{r^2}{w_1^2}\right). \tag{7}$$

The calculation of grad u in cylindrical coordinates is readily carried out from (4), and so we find

$$(\mathrm{grad}\, u)\cdot dS = -(-1)^m k\frac{w_0}{w_1}\left(1 + \frac{r^2}{R_1^2}\right) \cdot \exp\left(-\frac{r^2}{w_1^2}\right) r\, dr\, d\phi. \tag{8}$$

Using (4), (7), and (8), and ignoring terms of order $1/k_0^2w_0^2$ in comparison with unity, the integrals in (3) can be evaluated to give

$$\frac{\delta f}{f} \doteq -\frac{2}{k_0^4w_0^4}\left(1 - \frac{d}{2R_1}\right) \tag{9}$$

where k_0 is an eigenvalue of a symmetric mode. Note that $R_1 = kw_1^2/ad$ in Erickson's notation. Comparison with (2) shows that the error due to finite E_x on the mirror surfaces is of the same order as that due to the approximation made in the wave equation.

As a check on (9), we may use the inequality II.5, with II.7, of [2]. On putting $r = w_1$ in (7) to find the maximum value of v_s; i.e., $|\delta u|_{\max}$ in the notation of [2], we find

$$\frac{\delta f}{f} \le \frac{16e^{-1}}{k_0^4w_0^4}\left(1 - \frac{d}{2R_1}\right) \tag{10}$$

which verifies (9) in order of magnitude.

It is important to note that (9) depends on the assumption that E_y may be neglected in using the divergence equation; this is intuitively plausible, but (9) cannot be regarded as more than a reasonable estimate of the error resulting from failure to satisfy the boundary conditions.

It does seem, however, that an error of the order of $1/k_0^4w_0^4$ still remains after Erickson's corrections have been applied. Nevertheless, the agreement he obtains between theory and experiment is so impressive that it seems likely that the additional error is independent (or almost independent) of the radial mode number. It would be interesting to extend the present calculations to study this possibility.

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